CP–75–8

ANALYSIS AND COMPUTATION OF EQUILIBRIA AND REGIONS OF STABILITY

With Applications in Chemistry, Climatology, Ecology, and Economics

RECORD OF A WORKSHOP

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Immunity - A Mathematical Model

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(presented July 30)

I. Assumptions

Immunity may be described by two variables:

x - number of bacterias

y - number of lymphoides

Three main processes are suggested:

A - reproduction of the bacterias

B - production of the lymphoides

C - destruction of both the lymphoides and the bacterias as the result of the interaction.

The differential equation:

$$\frac{dx}{dt} = A - C$$

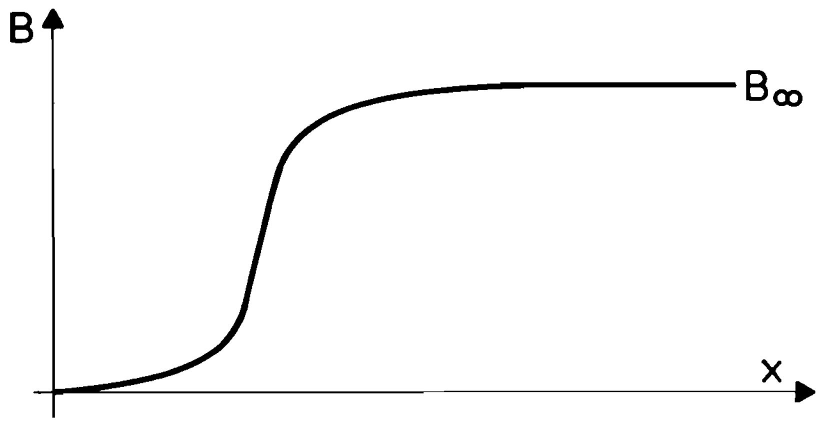
$$\frac{dy}{dt} = B - C \quad .$$

II. A more concrete model

$A = \alpha x$ exponential growth of the bacterias

$C = \gamma y$ exponential distruction of the lymphoides

$B = B(x)$ variable level of the immunity-defense



The typical behavior of the immunity-defense depends on the number of bacterias according to the physiological principle "all or nothing."

Remarks for biologists

More realistic is a step-by-step inclusion of the various levels of immunity. The number of levels ("barriers") is large, maybe seven or more. The main properties, however, of the immunity-process in time is clear in the simple case of the one-barrier immunity.

III. Model

Variables $x \to \frac{x}{x_o}$, $y \to \frac{y}{y_o}$, $t \to \frac{t}{t_o}$

$$\frac{dx}{dt} = \alpha x - y$$

$$\frac{dy}{dt} = \beta(x) - y \quad .$$

Remarks for mathematicians

The system is strictly equivalent to the one describing the mechanical motion with viscosity.

Proof

$$\frac{d^2x}{dt^2} + (1 - \alpha)\frac{dx}{dt} + f(x) = 0 \quad .$$

Here

$$f(x) = \beta(x) - \alpha x = -\frac{\partial U}{\partial x} \quad ,$$

$$U = \int_{x_0}^{x} [\beta(x) - \alpha x]dx$$ - potential function of the mechanical system

In the whole (x,y)-space the system has the Chetaev function (the generalization of the Ljapunov function).

If

$$N = \frac{(\alpha x - y)^2}{2} + U(x) \quad ,$$

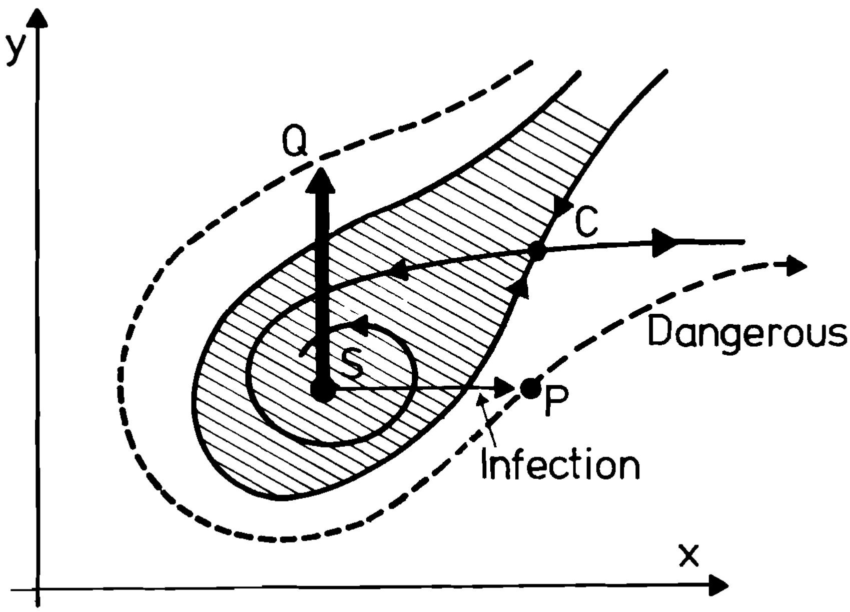
then

$$\frac{dN}{dt} \equiv (\alpha - 1)(\alpha x - y)^2 \quad .$$

All results about stability can be deduced from this function.

IV. Results and Phase Portraits

Weak (non-sterile) immunity



Possible result of uncontrolled use of antibiotics

S - point of non-sterile immunity (stable)

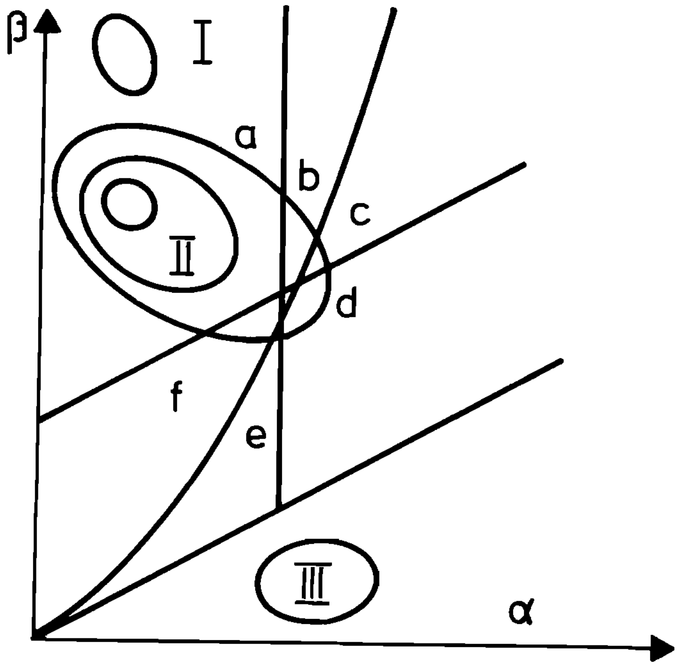
C - critical point (saddle, non-stable)

P - initial point after infection

Q - initial point after use of antibiotics

V. Results

Tuberculosis immunity structures of the different populations



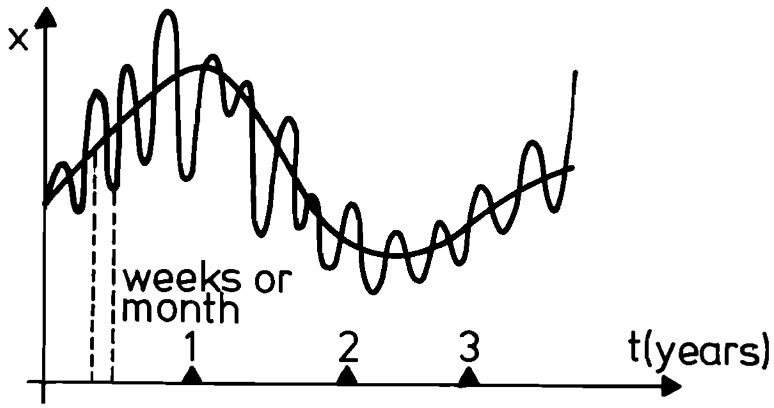
I - Rats Very strong immunity, resistence.

II - Human population "Model-treated" WHO data; various types of immunity, more than 6(a, b, c, d, e, f), maybe ~ 100.

III - Guinea pigs No immunity.

VI. Future study

In the model presented the rapid processes are omitted. They may, however, be very important in critical situations.



The real time-dependence

The model presents the "average" description only.

A more precise model must be constructed. The process B production of the lymphoides depends essentially on the rapid variables also.

Therefore:

$$\frac{dx}{dt} = dx - y$$

$$\beta(x) = b(x,u,v)$$

$$\frac{dy}{dt} = b(x,u,v) - y$$

$$\varepsilon \frac{du}{dt} = f(s,y,u,v) \qquad \varepsilon \text{ small parameter}$$

$$\varepsilon \frac{dv}{dt} = g(x,y,u,v) \qquad \varepsilon \sim \frac{\text{"week"}}{\text{"year"}} \sim \frac{1}{50} \quad .$$

What are the rapid variables? They probably describe energy processes in the whole organism. Some biologists believe (I also) that the energy processes are connected closely with stress-events.